B.Sc. Semester-III Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 32111 **Course Code : SH/MTH/301/C-5**

Course Title : Theory of Real Functions & Introduction to Metric Space

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meanings.

1. Answer any **five** questions: 2×5=10

a) Verify that on the curve $y=px^2+qx+r$, the chord joining the points for which x=a, x=b is parallel to the tangent at $x=\frac{a+b}{2}$.

b) Evaluate $\lim_{x\to 0^+} x\left[\frac{1}{x}\right]$.

c) State Caratheodory's theorem.

d) Find the value of c of Lagrange's MVT for the function $f(x)=2x^2+3x+4$ in [1,2].

e) Let $f(x)=\begin{cases} ax^2+b, & x<-1 \\ bx^2+ax+4, & x\geq -1 \end{cases}$. If $f'(x)$ exists for all x, find the values of a and b.

f) Verify Rolle's theorem for the function

$$f(x)=\log\left[\frac{x^2+ab}{(a+b)x}\right] \text{ in } [a, b].$$

g) Let (X, d) be a metric space. Define $\rho: X\times X\to\mathbb{R}$ by $\rho(x, y)=\frac{d(x, y)}{1+d(x, y)}$, $\forall x, y\in X$. Show that ρ is a metric on X.

h) Let d be the discrete metric defined on the set of real numbers $\mathbb{R}$. Examine whether the metric space $(\mathbb{R}, d)$ is separable or not.

2. Answer any **four** questions: 5×4=20

a) i) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable with $f(1)=6$ and $f'(x)\geq 2$ for $1\leq x\leq 4$. What is the minimum possible value of $f(4)$? 3

ii) Find values of the constants a, b and c for which the graphs of the two functions $f(x)=x^2+ax+b$ and $g(x)=x^3-c$, $x\in\mathbb{R}$, intersect at the point (1,2) and have the same tangent there. 2

b) i) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. If for all $x \in Q,\ f(x) = g(x)$ then prove that $f = g$. 3

ii) If $f(x) = \begin{cases} \dfrac{x\left(e^{\frac{1}{x}} - e^{-\frac{1}{x}}\right)}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then show that f is not derivable at $x = 0$. 2

c) i) Find the maximum and minimum values of $x + \sin 2x$ for $0 < x < 2\pi$. 3

ii) Show that the function $f(x) = \dfrac{1}{1+x^2},\ x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. 2

d) State and prove Cauchy's mean value theorem. 2+3

e) If $f(a+h) = f(a) + hf'(a) + \dfrac{h^2}{2!}f''(a) + \dfrac{h^3}{3!}f'''(a+\theta h)$ and if $f^{iv}(x)$ is continuous and non-zero at $x = a$ then prove that $\lim\limits_{h \to 0} \theta = \dfrac{1}{4}$. 5

f) Let a function f be continuous on $[a, b]$ and $f''(x)$ exist finitely for all $x \in (a, b)$. If the line segment joining the points $P(a, f(a))$ and $Q(b, f(b))$ intersects the curve of f at some point R other than P and Q, then prove that $f''(c) = 0$ for some $c \in (a, b)$. 5

3. Answer any **one** question: $10 \times 1 = 10$

a) i) Let a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f''(x)$ exist for all $x \in (a, b)$. If $f(a) = f(b) = 0$ and $a < c < b$, then show that there exists a point $p \in (a, b)$ such that

$$f(p) = \frac{1}{2}(c-a)(c-b)f''(p).$$ 4

ii) Prove that $\dfrac{2x}{\pi} < \sin x < x$ for $0 < x < \dfrac{\pi}{2}$. 3

iii) In a metric space (X, d), show that for all $x, x', y, y' \in X,\ |d(x, y) - d(x', y')| \leq d(x, x') + d(y, y')$. 3

b) i) Use Taylor's theorem to prove that $x - \dfrac{x^2}{2} < \log(1+x) < x$ for $x > 0$. 4

ii) In $C[0, 1]$, determine the values of $d_\infty(f, g)$ and $d_1(f, g)$ where $f(x) = x^3 + x + 1$ and $g(x) = x^3 + x^2 + \dfrac{1}{2}x + 1$, where $d_\infty(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|$,

$d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx$. 4

iii) Give an example of a function $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, such that $f'(x) = 0$ for all $x \in D$ but f is not a constant function. 2

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